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# MEROMORPHIC SOLUTIONS OF THE EQUATIONS OF MOTION OF A HEAVY SOLID WITH A FIXED POINT†

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All the solutions of the problem of the exponents of the first terms of the Laurent series which satisfy the equations of motion of a heavy solid with a fixed point are found using Newton's polyhedron method. Of the twenty-six solutions obtained seventeen are new. The conditions for arbitrary constants of integration to appear in the Laurent series are investigated.

The problem of the meromorphic solutions of the equations of motion of a heavy solid with a fixed point was formulated for the first time in [1] and investigated in [2, 3] by Kowalevski. In the Euler and Lagrange cases known previously the solutions of the equations of motion were expressed in terms of elliptic functions of time, i.e. functions that could be expanded in Laurent series and which have no other singularities apart from poles. Kowalevski raised a question of investigating in what other cases, apart from existing ones, the solutions of the equations of motion in this problem possess the same properties. We are primarily interested in solutions which contain a complete set of constants of integration, in this case five. She takes the series (the summation with respect to $n$ is from zero to infinity)

$$\begin{gathered} p=\tau^{n_1}\Sigma p_n\tau^n, \quad q=\tau^{n_2}\Sigma q_n\tau^n, \quad r=\tau^{n_3}\Sigma r_n\tau^n \\ \gamma_1=\tau^{m_1}\Sigma f_n\tau^n, \quad \gamma_2=\tau^{m_2}\Sigma g_n\tau^n, \quad \gamma_3=\tau^{m_3}\Sigma h_n\tau^n \end{gathered} \tag{0.1}$$

where $n_i$ and $m_i$ $(i=1, 2, 3)$ are negative integers, and asserts [1] that it follows from a comparison of the exponents of the first terms of the series that

$$n_i=-1, \ m_i=-2 \ \ (i=1,2,3) \tag{0.2}$$

without, however, considering the question of whether this system of values of the exponents is unique or not.

Markov [4] first drew attention to the incompleteness of the solution of the Kowalevski problem (see also [5, 6]).

*Markov's first objection*. One cannot conclude from a comparison of the exponents of the first terms of the Laurent series (0.1) that the values (0.2) are the only ones possible. If we consider the systems of exponents of the first terms of the Laurent series, we will have

$$\begin{gathered} n_1-1, \ n_2+n_3, \ m_3, \ m_2 \quad (1,2,3) \\ m_1-1, \ n_3+m_2, \ n_2+m_3 \quad (1,2,3) \end{gathered} \tag{0.3}$$

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In each of the six systems Kowalevski equates not two but all four or three numbers, and hence, neglects, without sufficient justification, an innumerable set of cases, such as, for example

$$n_i = -2, \ m_i = -4 \quad (i = 1, 2, 3)$$

*Markov's second objection.* Kowalevski ignores the case of multiple roots of its main determinant, but nevertheless, the possibility that a unique general integral also exists when there are multiple roots is not ruled out [4].

Appelrot [6] and Nekrasov [7] drew attention to Markov's observations. Appelrot generalized the system of exponents indicated by Markov, taking it in the form

$$n_i = -n, \ m_i = -2n \quad (i = 1, 2, 3) \tag{0.4}$$

and proved a theorem that the solutions of the equations of motion of a heavy solid can have a third-order pole. By considering the case of multiple roots Appelrot and Nekrasov found a special case, missed by Kowalevski, which had previously been found by Hess [8].

Lyapunov [9] took the second objection of Markov and proved the uniqueness of the general cases of Euler, Lagrange and Kowalevski integrability, in which the solutions of the equations of motion are expressed by unique functions of time. Lyapunov concludes [9]: "If, therefore, we can speak of some new case of the uniqueness of the functions $p, q, r, \gamma_1, \gamma_2, \gamma_3$ for real $A, B, C$ $x_0, y_0, z_0$ and for non-zero $A, B, C$, this can only be on the assumption that the initial values of these functions are subject to known conditions".

After the above papers were published it was assumed for a long time that the question of the unique solutions of the problem of the motion of a solid had been finally clarified [10]. Nevertheless, a number of papers on this problem appeared, where special cases of the integrability of the equations of motion were indicated, in which the solutions were unique functions of time. However, apart from the Hess–Appelrot case, no other special cases of the motion of a solid were obtained by Kowalevski's method, and in the well-known book by Golubev [11], it is merely mentioned.

Only in two papers by Bogoyavlenskii [12, 13] was it convincingly shown that Kowalevski's method can also be used to obtain special solutions containing less than five arbitrary constants of integration, and he succeeded in obtaining the new conditions for such solutions to exist and in finding the conditions which connected the constants of integration. It was quite correctly pointed out that the conditions for special cases to exist, when the solutions are meromorphic, and also the nature of the limitations imposed on the initial conditions, had not been investigated at all.

Note that Kowalevski's method has been successfully used in many other problems with a physical content, such as the Henon–Heiles system, Todd chains, etc. [5] (see also [14–17]).

The purpose of the present paper is to give a comprehensive answer to Markov's first objection.

1. We will consider the problem of the possible systems of exponents of the first terms of the Laurent series for the solutions of the problem of the motion of a heavy solid with a fixed point. To do this we will use the method of greatest and least exponents or, as it is now called, Newton's polyhedron method [18] (we recall Markov's objection regarding this method). By the well-known method in [19], we must take the expressions for the first terms of the Laurent series (0.1) and substitute into the system of equations of motion of the heavy solid

$$A\dot{p} + (C - B)qr = Mg(y_0\gamma_3 - z_0\gamma_2), \quad \dot{\gamma}_1 = r\gamma_2 - q\gamma_3, \quad (A, B, C) \ (1, 2, 3) \ (x_0, y_0, z_0) \tag{1.1}$$

The system of equations (1.1), as we know, has three first algebraic integrals (energy, momentum and geometrical)

$$\begin{gathered} Ap^2 + Bq^2 + Cr^2 = 2Mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) + h \\ Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = l, \quad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \end{gathered} \tag{1.2}$$

The system of exponents has the form (0.3). Comparing the four exponents in the first three equations and the three exponents in the last three equations we have

$$n_1 - 1 = n_2 + n_3 = m_3 = m_2 \quad (1,2,3)$$
$$m_1 - 1 = n_3 + m_2 = n_2 + m_3 \quad (1,2,3)$$

It can be shown that the system of exponents (0.2), indicated by Kowalevski, serves as a unique solution for these equations. But to obtain the Laurent series with non-zero constant coefficients $p_0$, $q_0$, $r_0$, $f_0$, $g_0$ and $h_0$ it is sufficient to compare not four and three exponents, as Kowalevski did, but two. Comparing the exponents in (0.3) in pairs, we obtain the following equations

$$n_1 - 1 = n_2 + n_3, \quad n_1 - 1 = m_3, \quad n_1 - 1 = m_2, \quad n_2 + n_3 = m_3,$$
$$n_2 + n_3 = m_2, \quad m_3 = m_2 \quad (1,2,3)$$
$$m_1 - 1 = n_3 + n_2, \quad m_1 - 1 = n_2 + m_3, \quad n_3 + m_2 = n_2 + m_3 \quad (1,2,3) \tag{1.3}$$

In order to obtain some solution for the numbers $n_i$, $m_i$, we must take from each row (from all six) one equation and combine them in the system. The solutions of each such system gives the necessary set of numbers $n_i$, $m_i$. Since the number of different triples of equations obtained from the first three rows (1.3) is 216, and from the second three rows is 27, the number of all possible sextets of equations, which exhaust the possible sets of numbers $n_i$, $m_i$ is equal to $216 \times 27 = 5832$. This is obviously the "innumerable set of other cases", which Markov had in mind.

Analysing Eqs (1.3) we note that in the fourth, fifth and sixth rows the last equations are dependent, and hence many solutions are possible in which one of the quantities $n_i$, $m_i$ is ambiguous.

All the systems obtained in this way were solved using the "REDUCE" software package. Many solutions were obtained in which one or several of the quantities $n_i$, $m_i$ are arbitrary. A large number of these solutions are the Kowalevski and Appelrot solutions (0.3) and (0.4).

We must choose from the number of solutions obtained those which satisfy the principle of greatest and least exponents.

The solutions of the systems obtained were checked to correspond to this principle and were chosen in such a way that in each row in (0.3) there were two equal numbers and, in addition, these two equal numbers must be the least. Laurent solutions were obtained in this case arranged in increasing powers of the time $\tau$. An analysis of all possible systems of solutions enabled us to distinguish the following independent systems of solutions which satisfy the principle of least exponents and give comprehensive answers to Markov's first objection

1. $n_1 = -3 - m_3, \quad n_2 = -3 - m_3, \quad n_3 = -1, \quad m_1 = m_2 = -2, \quad m_3 > -2, \quad A = B$
2. $n_1 = m_3 + 1, \quad n_2 = m_3 + 1, \quad n_3 = -1, \quad m_1 = m_2 = -2, \quad m_3 > -2, \quad z_0 = 0$
3. $n_1 = n_2 = n, \quad n_3 = -1, \quad m_1 = m_2 = -2, \quad n < -1, \quad A \doteq B, \quad m_3 > -2$
4. $n_1 = n_2 = n_3 = -1, \quad m_2 = m_3 = -2, \quad m_1 > -2$
5. $n_1 = n_2 = m_2 / 2, \quad n_3 = -1, \quad m_1 = m_2, \quad m_3 = m_2 / 2 - 1, \quad m_2 < -2, \quad z_0 = 0$
6. $n_1 = n_2 = m_2 / 2, \quad n_3 = -1, \quad m_1 = m_2, \quad z_0 = 0, \quad m_2 < -2, \quad m_3 > -2$
7. $n_1 = n_2 = n_3 = -1, \quad m_1 = -2, \quad m_2 = m_3 = m, \quad m < -2, \quad x_0 = 0$
8. $n_1 = n_2 = n_3 = -1, \quad m_2 = m_3 = m, \quad m > -2, \quad m_1 > -2, \quad m_1 > m$
9. $n_1 = m_1 + 1, \quad n_2 = m_1 + 1, \quad n_3 = -1, \quad m_1 = m_2, \quad A = B, \quad m_1 < -2, \quad m_3 > -2$
10. $n_1 = m_1 + 1, \quad n_2 = m_1 + 1, \quad n_3 = -1, \quad m_1 = m_2, \quad m_3 = -2, \quad m_1 < -2, \quad A = B$

11. $n_1 = (2m_1 + 1)/3,\ \ n_2 = n_3 = (m_1 - 1)/3,\ \ m_2 = m_3,\ \ m_3 = (2m_1 - 2)/3,\ \ m_1 < -2$

12. $n_1 = -1,\ \ n_2 = m_1 + 1,\ \ m_2 = m_3 = -2,\ \ m_1 + n_3 > -3,\ \ m_1 < n_3 - 1,\ \ x_0 = 0,\ \ m_1 > -2$

13. $n_1 = n_2 = -1,\ \ m_1 = m_2 = m_3 = -2,\ \ n_3 > 1$

14. $n_1 = n_2 = n_3 = -1,\ \ m_1 = m_2 = m_3 = \pm 1$

15. $n_1 = n_2 = m_2 - m_3 - 1,\ \ n_3 = -1,\ \ m_1 = m_2,\ \ A = B,\ \ m_3 > -2,\ \ m_3 > m_2,\ \ m_2 < -2$

16. $n_1 = n_2 = m_1/2,\ \ n_3 = m_3 - m_1/2,\ \ m_1 = m_2,\ \ m_3 > (3/2)m_1 + 1,\ \ m_3 < m_1/2 - 1$

17. $n_1 = n_2 = n,\ \ n_3 = -1,\ \ m_1 = m_2 = 2n,\ \ n < -1,\ \ m_3 > -2,\ \ z_0 = 0$

18. $n_1 = n_3 = -1,\ \ n_2 = n,\ \ m_1 = m_2 = m_3 = m,\ \ n > -1,\ \ m < -2$

19. $n_1 = n_3 = n,\ \ n_2 = m,\ \ m_1 = m_3 = 2n,\ \ m_2 = m + n,\ \ y_0 = 0,\ \ n_2 < -1,\ \ n_3 < -1,\ \ n_2 < n_3$

20. $n_1 = n_2 = n_3 = n,\ \ m_1 = m_2 = m_3 = 2n$

21. $n_1 = n_2 = m_1 + 1,\ \ n_3 = -1,\ \ m_1 = 2m + 2,\ \ m_2 = 2m + 2,\ \ m_3 = m,\ \ z_0 = 0,\ \ m < -2$

22. $n_1 = n_2 = m_3 + 1,\ \ n_3 = -1,\ \ m_1 = m_2 = m,\ \ m < -2,\ \ m_3 < m,\ \ m < 2m_3 + 2$

23. $n_1 = m + 1,\ \ n_2 = n_3 = m/2,\ \ m_1 = (3/2)m + 1,\ \ m_2 = m_3 = m,\ \ m < -2$

24. $n_1 = n_2 = m/4 - 1/2,\ \ n_3 = m/2,\ \ m_1 = m_2 = 3m/4 - 1/2,\ \ m_3 = m,\ \ m < -2$

25. $n_1 = -1,\ \ n_2 = n_3 = m - n - 1,\ \ m_1 = n,\ \ m_2 = m_3 = m,\ \ m < -2,\ \ n > m,\ \ B = C,\ \ n > -2$

26. $n_1 = n_2 = n_3 = n,\ \ m_1 = m_2 = m_3 = m,\ \ n < -1,\ \ m < 2n$

Solutions 2, 4, 8, 13 and 14 were obtained previously in [20]. All the remaining solutions require checking. We will not indicate Kowalevski's solutions here. The system of solutions 20 corresponds to Appelrot's solutions.

**2**. The next stage of the investigation involves obtaining the coefficients of the first terms of the Laurent series for each version of the Kowalevski exponents from the system of non-linear algebraic equations. Then, all the remaining coefficients of these series are obtained successively from the systems of linear equations. In addition, it is necessary to obtain the limitations on the parameters of the problem, on satisfying which arbitrary constants appear in the coefficients of the Laurent series.

In view of the fact that a detailed description of these investigations would take up a large amount of space, we will confine ourselves to stating the results obtained. Since many solutions for the Kowalevski exponents obtained in Section 1 contain undetermined quantities, we will confine ourselves to one example of the construction of the Laurent series giving an undetermined quantity. It is also possible to construct series for other values of the undetermined quantities.

The two known solutions of the problem of the Kowalevski exponents (0.2) and (0.4) will not be investigated further.

Using the example of solution 1 for the Kowalevski exponents we will describe how systems of algebraic equations for finding the first and subsequent coefficients of the Laurent series can be obtained. We have

$$n_1 = n_2 = -3 - m_3,\ \ n_3 = -1,\ \ m_1 = m_2 = -2,\ \ m_3 > -2,\ \ A = B$$

We will write the system of exponents of the first terms of the Laurent series for the whole system (0.1). In this case it is as follows:

$$\begin{array}{l} -4 - m_3,\ \ -4 - m_3,\ \ \{m_3, -2\} \\ -4 - m_3,\ \ -4 - m_3,\ \ \{-2, m_3\} \\ -2,\ \ \{0\},\ \ -2,\ \ -2 \\ -3,\ \ -3,\ \ -3 \end{array} \tag{2.1}$$

$$-3,\ -3,\ -3$$
$$\{m_3-1\},\ -5-m_3,\ -5-m_3$$

It is obvious that in each row there are equal exponents, and when $m_3>-2$ this system satisfies the principle of least exponents. To set up a system of non-linear algebraic equations, which we must do in order to obtain the coefficients of the first terms of the Laurent series, we choose from system (0.1) only those terms whose exponents in (2.1) are least. In (2.1) these are all the exponents not within the braces. Finally, the system of non-linear algebraic equations for obtaining the coefficients of the first term of the Laurent series has the form

$$\begin{aligned} &An_1p_0+(C-A)q_0r_0=0, \quad && m_1f_0=r_0g_0-q_0h_0 \\ &An_2q_0+(A-C)p_0r_0=0, \quad && m_2g_0=p_0h_0-r_0f_0 \\ &Cn_3r_0=Mg(x_0g_0-y_0f_0), \quad && 0=q_0h_0-p_0g_0 \end{aligned} \tag{2.2}$$

It has two solutions

1. $p_0=0,\ q_0=0,\ r_0=\pm 2i,\ f_0=\pm 2C/[Mg(x_0\pm iy_0)],\ i=\sqrt{-1}$
$g_0=\mp 2Ci/[Mg(x_0\pm iy_0)],\ h_0$ is an arbitrary number

2. $p_0=\pm iq_0,\ r_0=\mu i,\ f_0=\mu,\ g_0=\mu_1 i,\ h_0=\mu_2 i/q_0$, where

$$\begin{aligned} &\mu=A(3+m_3)/(C-A),\quad \mu_1=-CA(3+m_3)/[Mg(C-A)(ix_0-y_0)] \\ &\mu_2=-CA[2(C-A)-A(3+m_3)](3+m_3)/[Mg(ix_0-y_0)(C-A)^2] \end{aligned} \tag{2.3}$$

The abbreviated system of linear algebraic equations for finding the further coefficients of the Laurent series has the form

$$\begin{aligned} &(n+n_1)Ap_n+(C-A)(q_0r_n+r_0q_n)=P_m \\ &(n+n_2)Aq_n+(A-C)(r_0p_n+p_0r_n)=Q_m \\ &(n+n_3)Cr_n-Mg(y_0f_n-x_0g_n)=R_m \\ &(n+m_1)f_n-r_0g_n+g_0h_n-g_0r_n+h_0g_n=F_m \\ &(n+m_2)g_n-p_0h_n+r_0f_n-h_0p_n+f_0r_n=G_m \\ &-q_0f_n+p_0g_n-f_0q_n+g_0p_n=H_m,\quad n>m \end{aligned} \tag{2.4}$$

where $P_m, Q_m, \ldots, H_m$ are polynomials of the coefficients of the Laurent series obtained in the preceding steps.

The system of equations (2.4) differs from the system considered by Kowalevski. This system contains only the coefficients of those terms of the Laurent series, the sum of the exponents of which is least for this step. We will further show that in solutions 1 and 2 the determinant of system (2.4) is identically zero at any step. In order to construct Laurent series in this case we must require that the condition for the system of equations (2.4) to be compatible is satisfied. It has been established that for solution 1 in (2.3) this condition of compatibility is not satisfied for any step. For solution 2, from the condition for system (2.4) to be compatible it reduces to the equation for finding one coefficient $h_i$. Here we obtain a Laurent series containing two constants. For the other constants to occur in the coefficients of the series it is necessary for some fifth-order minors to vanish. These minors are obtained by cancelling some columns and rows of the determinant of system (2.4). In all we can form 36 different minors. We will consider the fifth-order minors to be identically non-zero.

Cancelling the first column and the first row we obtain the following determinant

$$D_1 = (n-4)(y_0 g_0 + f_0 x_0) f_0 (A - C) \tag{2.5}$$

It vanishes when: (1) $A = C$ (kinetic symmetry), (2) $x_0 = y_0 = 0$ (the Lagrange case), and (3) $n = 4$.

In case 3 a constant of integration occurs in the coefficients of the series at the fourth step. The condition for the system of five linear equations to be compatible leads to the fixing of the constant $q_0$, but a new constant occurs.

Cancelling the first column and the fourth row we obtain the determinant

$$D_2 = \{(A-C)^2 p_0 r_0 + (A-C)(n+n_4)Aq_0\}\{p_0 y_0 - q_0 x_0\} \tag{2.6}$$

The equation $D_2 = 0$ has a single root

$$n = 6 + 2m_3 \tag{2.7}$$

The determinant obtained by cancelling the first column and the last row is

$$D_3 = \{n - (6 + 2m_3)\}\{y_0(ir_0 + 2 - n) - x_0[(n-2)i - r_0]\} \tag{2.8}$$

If $x_0 = 0$ in (2.8), the equation $D_3 = 0$ has two roots

$$n = 6 + 2m_3, \quad n = 2 + A(3 + m_3)/(A - C)$$

The second root must be integer, which imposes limitations on the parameters of the problem. The triangle inequality will be satisfied if the inequalities $0 < C/A < 2$ are satisfied. If $C/A = 1/3$, we have $n = 5$, $m_3 = -1$; $n = 8$, $m_3 = 1$; $n = 11$, $m_3 = 3$; etc. If $A/C = 2/3$, we have $n = 8$, $m_3 = -1$; $n = 14$, $m_3 = 1$; $n = 20$, $m_3 = 3$, etc.

Cancelling the second column and the fourth row we obtain the determinants

$$D_4 = [n - (6 + 2m_3)][p_0 y_0 + q_0 x_0]$$

The equation $D_4 = 0$ has the root (2.7).

Cancelling the fourth column and the first row we obtain the determinants

$$D_5 = C(n-4)(n+n_3)\{(n+n_2)Ag_0 + f_0 r_0 (A - C)\}$$

The equation $D_5 = 0$ has the roots

$$n = 1, \; n = 0, \; n = 6 + 2m_3$$

Cancelling the last column and row we obtain a determinant with a single real root (2.7).

Finally, using solution 2 we can construct Laurent series containing two constants of integration.

The relation between the constants of the energy and momentum integrals can be found by substituting, for example, the Laurent series (0.1) into the integrals (1.2). We thereby obtain the following equations

$$A(p_2^2+q_2^2+2p_4p_0+2p_1p_3+2q_0q_4)+C(r_1^2+2r_0r_2)=2Mg(x_0f_2+y_0g_2+z_0h_1)+h$$
$$A(p_2f_2+p_0f_4+p_4f_0+p_1f_3+p_3f_1)+A(q_2g_2+q_0g_4+q_4g_0+q_1g_3+q_3g_1)+$$
$$+C(r_1h_1+r_0h_2+r_2h_0)=l$$
$$f_2^2+g_2^2+h_1^2+2h_0h_2+2g_0g_4+2g_1g_3+2f_1f_3+2f_0f_4=1$$

Since all the coefficients of the series are expressed in terms of arbitrary constants, these three relations also contain these constants. Eliminating them, for example, using the result, we obtain equations relating the constants of integration $h$ and $l$.

In solution 2

$$n_1=n_2=m_3+1,\quad n_3=-1,\quad m_1=m_2=-2,\quad z_0=0,\quad m_3>-2$$

Here it is possible to construct Laurent series for the following conditions:

1. $m_3=0$, $A=B$, $C=A/2$, $x_0=0$, $z_0=0$, the constant of integration occurs for $n=1$;
2. $m_3=1$, $C=[21-\sqrt{(133)}]A/8$, $B=[\sqrt{(133)}-9]A/4$, $x_0=z_0=0$, the constant of integration occurs for $n=1$, and the triangle inequalities for the moments of inertia are satisfied;
3. $m_3=0$, $A=2(B-C)$, $B=24C/(5+\sqrt{(288)})$, $x_0=z_0=0$, the constant of integration occurs for $n=3$, and the triangle inequalities are satisfied.

By changing the value of $m_3$ we can obtain other conditions for the Laurent series to exist. In all the cases mentioned above we have series containing two arbitrary constants.

In solution 3

$$n_1=n_2=n_0,\quad n_3=-1,\quad m_1=m_2=-2,\quad m_3>-2,\quad A=B$$

If $n_0=-2$, $-3$ and $-4$, we have $C=2A$, $5A/2$, $3A$, etc.

Here it is possible to construct Laurent series containing two arbitrary constants for different $n_0$ and $m_3$, for example, for $n_0=-2$ and $m_3=0$.

In solution 4

$$n_1=n_2=n_3=-1,\quad m_2=m_3=-2,\quad m_1>-2$$

The constants of integration occur in the Laurent series at the second step, if the following conditions are satisfied

$$BC=4(A-B)(A-C)\quad\text{or}\quad C=2(A-B),\ m_1=1$$

and at the fourth step, if the following conditions are satisfied

$$9BC=4(A-B)(A-C)\quad\text{or}\quad 3C=2(A-B),\ m_1=3$$

The coefficients of the Laurent series contain three arbitrary constants if $z_0=0$, and two arbitrary constants if $f_0=0$.

In solution 5

$$n_1=n_2=m_2/2,\quad n_3-1,\quad z_0=0,\quad m_3=m_2/2-1,\quad m_2<-2,\quad m_1=m_2$$

When the following conditions are satisfied

$$C=(B^2+2A^2-3AB)/(A-B),\quad 2(B-C)Ax_0^2-(2B^2-A^2-3BC+AB+AC)y_0^2=0$$

the Laurent series contain two arbitrary constants.

In solution 6

$$n_1=n_2=m_2/2,\quad n_3=-1,\quad z_0=0,\quad m_2<-2,\quad m_3>-2$$

The Laurent series contain three arbitrary constants when the following conditions are satisfied

$$m_3=-1,\quad m_2=-4$$
$$A=B=3C/5\ (n=4);\quad A=B=4C/7\ (n=6);\quad A=B=5C/9\ (n=8);$$
$$A=B=6C/11\ (n=10)$$

etc. It is also possible to construct series for other combinations of $m_2$ and $m_3$.

In solution 7

$$n_1=n_2=n_3=-1,\quad m_1=-2,\quad m_2=m_3=m,\quad m<-2,\quad x_0=0$$

Provided that

$$n=Cz_0^2/[(B-C)y_0^2]+B/(B-C)-m$$

is a positive integer, it is possible to construct Laurent series with two arbitrary constants. The possibility of constructing series with $m=-3$ has been shown. The constant of integration occurs when $n=2$.

In solution 8

$$n_1=n_2=n_3=-1,\quad m_2=m_3=m,\quad m_1>m>-2$$

When

$$m=-1,\quad m_1=0\quad x_0(B+C)\sqrt{(B+C)(B-C)}+2z_0BC=0,\quad y_0=-z_0,\quad A=B+C$$

it is possible to construct Laurent series with two constants of integration.

In solution 9

$$n_1=n_2=m_1+1,\quad n_3=-1,\quad m_1=m_2,\quad m_1<-2,\quad m_3>-2,\quad A=B$$

If $n=A(m_1+1)/(A-C-m_1)$ is a positive integer we have Laurent series with three constants of integration if the following conditions are satisfied

$$m_1=-3,\quad m_3=-1,\quad n=6,\quad 5A=3C$$

and the triangle inequalities are satisfied.

In solution 10

$$n_1=n_2=m_1+1,\quad m_1=m_2,\quad A=B,\quad m_3=-2,\quad n_3=-1,\quad m_1<-2$$

Provided that $m_1=-3$, $m_2=-2$ and $n=A(m_1+1)/(A-C-m_1)$ is a positive integer, for example with $n=6$, and $5A=3C$ we have Laurent series with two constants of integration.

In solution 11

$$n_1 = (2m_1 + 1)/3, \quad n_2 = n_3 = (m_1 - 1)/3, \quad m_2 = m_3 = 2(m_1 - 1)/3, \quad m_1 < -2.$$

With $m_1 = -5$ we have Laurent series containing three arbitrary constants.

In solution 12

$$n_1 = -1, \quad n_2 = m_1 + 1, \quad m_2 = m_3 = -2, \quad m_1 > -2, \quad n_3 > -1, \quad m_1 < n_3 - 1, \quad x_0 = 0$$

For $n_3 = 1$, $m_1 = -1$, $n_2 = 0$, $A = 3B$, $C = 2(A - B)$, $y_0 = 0$ we have Laurent series containing two arbitrary constants. Laurent series are also possible for other values of $m_1$ and $n_3$.

In solution 13

$$n_1 = n_2 = -1, \quad m_2 = m_3 = m_1 = -2, \quad n_3 > -1$$

For $n_3 = 1$, $z_0 = 0$, $A = 2C$ we have Laurent series with three constants of integration.

In solution 14

$$n_1 = n_2 = n_3 = -1, \quad m_1 = m_2 = m_3 = \pm 1$$

The Kowalevski determinant for $m_i = -1$ $(i = 1, 2, 3)$ is

$$D = n^3(n-3)(n-1)(n-2)$$

For the case $m_i = +1$ $(i = 1, 2, 3)$

$$D = n^3(n-3)(n+1)(n+2)$$

In the first case, when the following condition is satisfied

$$\begin{aligned}
&\{2C(A + B - C)(B - C)(A - B)F(A, B, C) + \\
&+ B(2B - A - C)(A + C - B)(A + B - C)F(A, B, C) + \\
&+ BC(A - C - B)^2(C - A)G(A, B, C)\}x_0 - \\
&- \{2B^2C(A - B)(A + B - C)W(A, B, C) + \\
&+ B(2B - A - C)(A + B - C)^2AB^{1/2}W(A, B, C) + \\
&+ BC(A + C - B)(A + B - C)(C - A)(B - C)A^{1/2}\}y_0 + \\
&+ \{2B^2C(A - B)(A + C - B)G(A, B, C) - \\
&- 2CA(A + B - C)(A - B)(B - C)G(A, B, C)\}z_0 = 0 \\
&F(A, B, C) = \{ABC(A - B)(C - A)\}^{1/2}, G(A, B, C) = \{B(B - C)(C - A)\}^{1/2} \\
&W(A, B, C) = \{C(A - B)(B - C)\}^{1/2}
\end{aligned}$$

we have Laurent series with four constants of integration.

In the second case, when the following conditions are satisfied

1. $\{-(B - C)^2(C - A)[ABC(A - B)]^{1/2} + AB(C - B)(2A - B - C)[A(A - B)]^{1/2}\}x_0 +$
$+ \{A(C + B)(C - A) + (2C - 3A)(B - C)\}(A - B)[C(B - C)]^{1/2}z_0 = 0, \; y_0 = 0$

2. or $B = 3AC/(A + 2C)$, $y_0 = 0$ (the first condition (1) is also satisfied), we have Laurent series with three constants of integration.

In solution 15

$$n_1 = n_2 = m_1 - m_3 - 1, \quad n_3 = -1, \quad m_1 = m_2, \quad A = B, \quad m_3 > -2, \quad m_2 < -2$$

If $m_1 = m_2 = -3$ and $m_3 = -1$ the Laurent series contain two arbitrary constants.

In solution 16

$$n_1 = n_2 = m_1/2, \quad n_3 = m_3 - m_1/2, \quad m_1 = m_2, \quad 3m_1/2 + 1 < m_3 < m_1/2 - 1, \quad z_0 = 0$$

If $m_1 = -6$ and $m_3 = -5$ Laurent series occur with two constants of integration.

Solution 17 is equivalent to solution 6. Solution 18 leads to well-known cases. Solution 19 is equivalent to solution 16. Solution 20 was considered by Appelrot. Solution 21 is equivalent to solution 5. Solution 22 is equivalent to solution 9. Solution 23 is equivalent to solution 11.

In solution 24

$$n_1 = n_2 = m/4 - 1, \quad n_3 = m/2, \quad m_1 = m_2 = 3m/4 - 1/2, \quad m_3 = m, \quad m < -2, \quad x_0 = y_0 = 0$$

If $m = -6$, $n_1 = n_2 = -2$, $n_3 = -3$, $m_1 = m_2 = -5$ we have Laurent series with three constants of integration.

Solution 25 is equivalent to solution 15. Solution 26 reduces either to the case considered by Appelrot or to Euler's case.

We can finally conclude that of the 26 solutions of the problem of the exponents of the first terms of Laurent series which satisfy the equations of motion of a heavy solid with a fixed point, only 17 are new.

The equivalence of the solutions follows from the same non-linear systems of algebraic equations for determining the coefficients of the first terms of the Laurent series, and also from the same linear system of algebraic equations for determining all subsequent coefficients of the series.

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